A Utility-Based Congestion Control Scheme for Internet-Style Networks with Delay[1]

Tansu Alpcan[2] and Tamer Başar[2]
(alpcan, tbasar)@control.csl.uiuc.edu

***Abstract*— In this paper, we develop, analyze and implement a congestion control scheme obtained in a noncooperative game framework where each user's cost function is composed of a pricing function, proportional to the queueing delay experienced by the user, and a broad class of utility functions capturing the user demand for bandwidth. Using a network model based on fluid approximations and through a realistic modelling of queues, we establish the existence of a unique equilibrium as well as its global stability for a general network topology. We also provide sufficient conditions for system stability when there is a bottleneck link shared by multiple users experiencing non-negligible communication delays. Based on these theoretical foundations, we implement a window-based, end-to-end congestion control scheme, and simulate it in *ns-2* network simulator on various network topologies with sizable propagation delays.**

Methods Keywords: Control theory, Mathematical programming/optimization, Simulations, Economics.

***Index Terms*— Congestion control; Internet; noncooperative games; stability.**

I. INTRODUCTION

Game theory provides a natural framework for developing pricing and congestion control mechanisms for the Internet. Users on the network can be modeled as players in a congestion control game where they choose their strategies or in this case flow rates. Players are noncooperative in terms of their demands for network resources, and have no specific information on other users' strategies. A user's demand or utility for bandwidth is captured in a utility function, and may not be bounded. On the other hand, one can devise a pricing function, proportional to the bandwidth usage of a user, in order to preserve the network resources and to provide an incentive for the user to implement end-to-end congestion control [1]. A useful concept in this noncooperative congestion control game is the one of a Nash equilibrium [2] where each player minimizes his/her own cost (or maximize payoff) given all other players' strategies. There is rich literature on game theoretic analysis of flow control problems utilizing both cooperative [3] and noncooperative [4], [5], [6] frameworks. Congestion control schemes utilizing pricing schemes based on explicit feedback have been proposed by Kelly et al. [7], [8], Gibbens et al. [9], and subsequent studies have further elaborated on this approach following its basic principles [10], [11], [12].

Although the game theoretic approach provides a suitable framework for formulating and studying congestion and flow control problems in general networks, there are some inherent restrictions on implementable cost functions in the case of Internet-style networks. For example, the current structure of the Internet makes it difficult, if not impossible, for users to obtain detailed real time information on the state of the network and on other users. Therefore, users are bound to use indirect aggregate metrics that are available to them, such as packet drop rate and variations in the average round trip time (RTT) of packets in order to infer the current situation in the network. Packet drops, for example, are currently used by most widely deployed versions of TCP as an indication of congestion. In this paper, however, we propose and analyze a pricing and congestion control scheme based on variations in the queueing delay a user experiences. A similar approach has been suggested in a version of the transfer control protocol (TCP), known as TCP Vegas [13]. Although TCP Vegas is more efficient than a widely used version of TCP, TCP Reno [14], the suggested improvements are empirical and based on experimental studies. Another study by Mo and Walrand [12] also makes use of an approach similar to the one in this paper. However, it is based on fairness and pricing concepts of Kelly, and employs only a narrow set of utility functions in describing user demands.

The noncooperative congestion control game introduced in this paper is characterized by a cost function

[1]Research supported in part by the NSF grant CCR 00-85917. All correspondence should be forwarded to: Prof. Tamer Başar, Coordinated Science Laboratory, University of Illinois, 1308 West Main Street, Urbana, IL 61801-2307 USA. Tel: (217) 333-3607; Fax: (217) 244-1653; E-mail: tbasar@control.csl.uiuc.edu

[2]Coordinated Science Laboratory and Department of Electrical and Computer Engineering, University of Illinois, 1308 West Main Street, Urbana, IL 61801 USA.

for each user that is defined as the difference of pricing and utility functions. The pricing function is proportional to the queueing delay experienced by the user, whereas the utility function that quantifies the user demand for bandwidth belongs to a broad class of strictly increasing and strictly concave functions. Through a network model based on fluid approximations, and a realistic queueing model, we show the existence of a unique 'Nash' equilibrium, under the assumption that the effect of a user's flow on congestion cost is vanishingly small, especially if the number of users is large. Furthermore, we establish the global stability of the equilibrium under a general network topology. We also investigate stability of the system in a network with non-negligible propagation delays, and provide sufficient conditions for stability in the case of a bottleneck node with multiple users. Based on the theoretical foundations developed, we design a window-based, end-to-end congestion control scheme for Internet-style networks, which is TCP-friendly [15]. This congestion control scheme is then simulated in Network Simulator 2 (*ns-2*) over Internet protocol (IP) for various network topologies. The rest of the paper is organized as follows: The underlying network model and cost function are given in the next section. In Section III, the existence of a unique equilibrium and global stability of the system under a general network topology are established. Section IV generalizes the stability analysis of Section III to the case with delay, with a single bottleneck link. In Section V we provide a realistic implementation of the congestion control scheme for IP networks. Section VI includes simulation results, and is followed by the concluding remarks of Section VII.

II. The Model

A. The Network Model

We consider a general network model based on fluid approximations. Fluid models are widely used in addressing a variety of network control problems such as congestion control [12], [5], [16], routing [5], [6], and pricing [7], [3], [17]. The topology of the network is characterized by a set of nodes $\mathcal{N} = \{1, \ldots, N\}$ and a set of links $\mathcal{L} = \{1, \ldots, L\}$, connecting the nodes. In this network model, we make the natural assumption of *connectivity*, and let $\mathcal{M} := \{1, \ldots, M\}$ denote the set of active users. Each link $l \in \mathcal{L}$ has a fixed capacity $C_l > 0$, and an associated buffer size $b_l \geq 0$. For simplicity, each user is associated with a (unique) connection. Hence, the i^{th} $(i \in \mathcal{M})$ user corresponds to a unique connection between the source and destination nodes, $s_i, de_i \in \mathcal{N}$, and we denote the corresponding route (path), which is a subset of $\mathcal{L}$, by R_i. The nonnegative flow, x_i, sent by the i^{th} user over this path R_i satisfies the bounds $0 \leq x_i \leq x_{i,max}$. The upper bound, $x_{i,max}$, on the i^{th} user's flow rate may be a user specific physical limitation, and cannot exceed the minimum capacity of the links on the route, $\min_l\{C_l \;, \; l \in R_i\}$.

It is possible to define a routing matrix, $\mathbf{A}$, as in [7] that describes the relation between the set of routes $\mathcal{R}$ associated with the users (connections) and links $l :\in \mathcal{L}$:

$$A_{l,i} = \begin{cases} 1, & \text{if source i uses link } l \\ 0, & \text{if source i does not use link } l \end{cases}, \quad \begin{array}{l} i \in \mathcal{M} \; and \\ l \in \mathcal{L} \end{array} \tag{1}$$

Using the routing matrix $\mathbf{A}$, the capacity constraints of the links are given by

$$\mathbf{A}\mathbf{x} \leq \mathbf{C} \quad , \tag{2}$$

where $\mathbf{x}$ is the $(M \times 1)$ flow rate vector of the users and $\mathbf{C}$ is the $(L \times 1)$ link capacity vector. If the aggregate sending rate of users whose flows pass through link l exceeds the capacity, C_l, of the link then the arriving packets are queued (generally on a first-come first-serve basis) in the buffer, b_l, of the link with $b_{l,max}$ being the maximum buffer size. Let the total flow on link l be given by $\bar{x}_l := \sum_{i:l \in R_i} x_i$. Thus, the buffer level at link l evolves in accordance with the following

$$\dot{b}_l(t) = \begin{cases} [\bar{x}_l - C_l]^-, & \text{if } \; b_l(t) = b_{l,max} \\ \bar{x}_l - C_l, & \text{if } \; 0 < b_l(t) < b_{l,max} \\ [\bar{x}_l - C_l]^+, & \text{if } \; b_l(t) = 0 \end{cases} , \tag{3}$$

where $\dot{b}_l(t)$ denotes $(\partial b_l(t)/\partial t)$, $[.]^+$ represents the function $\max(. \, , 0)$, and $[.]^-$ represents the function $\min(. \, , 0)$.

B. The Cost (Objective) Function

An important indication of congestion for internet-style networks is the variation in queueing delay, d, which is defined as the difference between the actual delay experienced by a packet, d^a, and the fixed propagation delay of the connection, d^p. If the incoming flow rate to a router exceeds its capacity, packets are queued (generally on a first-come first-serve basis) in the existing buffer of the router, leading to an increase in the RTT of packets. Hence, RTT on a congested path is longer than the base RTT, which is defined as the sum of propagation and processing delays on the path of a packet. The queueing delay at the l^{th} link, d_l, is a nonlinear function of the excess flow on that link, and is given by

$$\dot{d}_l(\mathbf{x},t) = \begin{cases} \left[\frac{1}{C_l}(\bar{x}_l - C_l)\right]^-, & \text{if } d_l(t) = d_{l,max} \\ \frac{1}{C_l}(\bar{x}_l - C_l), & \text{if } 0 < d_l(t) < d_{l,max} \\ \left[\frac{1}{C_l}(\bar{x}_l - C_l)\right]^+, & \text{if } d_l(t) = 0 \end{cases}, \quad (4)$$

in accordance with the buffer model described in (3), with $d_{l,max}$ being the maximum possible queueing delay. Thus, the total queueing delay, D_i, a user experiences is the sum of queueing delays on its path, namely $D_i(\mathbf{x},t) = \sum_{l \in R_i} d_l(\mathbf{x},t)$, $i \in \mathcal{M}$.

Let us define a cost function for each user as the difference between pricing and utility functions. The pricing function of the i^{th} user is linear in x_i, and is proportional to the total queueing delay $D_i(t)$ of the user. The utility function $U_i(x_i)$ is assumed to be strictly increasing, differentiable, and strictly concave; it basically describes the user's demand for bandwidth. Accordingly, we make use of variations in RTT to devise a congestion control and pricing scheme. The cost (objective) function for the i^{th} user at time t is thus given by

$$J_i(\mathbf{x},t) = \alpha_i D_i(\mathbf{x},t)\, x_i - U_i(x_i)\ , \quad (5)$$

which s/he wishes to minimize. In accordance with this objective, we consider a simple dynamic model of the network game where each user changes his flow rate in proportion with the gradient of his cost function with respect to his flow rate, $\dot{x}_i = -\partial J_i(\mathbf{x})/\partial x_i$. Thus, the update algorithm for the i^{th} user is:

$$\dot{x}_i = \begin{cases} \left[\frac{dU_i(x_i)}{dx_i} - \alpha_i\, D_i(\mathbf{x},t)\right]^-, & \text{if } x_i = x_{i,max} \\ \frac{dU_i(x_i)}{dx_i} - \alpha_i\, D_i(\mathbf{x},t), & \text{if } 0 < x_i < x_{i,max} \\ \left[\frac{dU_i(x_i)}{dx_i} - \alpha_i\, D_i(\mathbf{x},t)\right]^+, & \text{if } x_i = 0 \end{cases}, \quad (6)$$

where the effect of the i^{th} user on the delay, $D_i(\mathbf{x},t)$, s/he experiences is ignored. This assumption can be justified for networks with a large number of users, where the effect of each user is vanishingly small. Furthermore, from a practical point of view, it is extremely difficult if not impossible for a user to estimate his/her own effect on queueing delay.

III. Stability Analysis

In this section, we analyze the stability of the system described by (4) and (6). First, we investigate the simple case of a single link with a single user in order to gain further insight to the system. [1] We then generalize the analysis to a single link with multiple users. Finally, we establish stability for a general network topology with multiple links and users.

[1] Admittedly, in this case the assumption of an individual user not affecting the delay on a link is violated, but still this exercise is useful for the later analysis on the multiple users case.

A. Stability for a Single Link with a Single User

For a single user on a single link, the equations describing the dynamics of the system consist of the user algorithm, which is a simplified version of (6), and queueing delay equation for a single user derived from (4). For the time being we ignore the effects of boundaries on the system:

$$\begin{aligned} \dot{x}(t) &= \frac{dU(x)}{dx} - \alpha d(x,t) \\ \dot{d}(t) &= \frac{x}{C} - 1 \end{aligned}\ , \quad (7)$$

where d is the queueing delay, x is the user flow rate, and C is the link capacity.

The system (7) has a unique equilibrium point (x^*, d^*) given by $x^* = C$ and $d^* = (1/\alpha)\, dU(x^*)/dx$. Defining the queueing delay and flow rate around the equilibrium point, $\tilde{x} := x - x^*$ and $\tilde{d} := d - d^*$, we obtain the following equivalent system around the equilibrium:

$$\begin{aligned} \dot{\tilde{x}}(t) &= g(\tilde{x}) - \alpha \tilde{d}(t) \\ \dot{\tilde{d}}(t) &= \tfrac{1}{C}\tilde{x} \end{aligned}\ , \quad (8)$$

where the function $g(\tilde{x})$ is defined as

$$g(\tilde{x}) := \frac{dU(x)}{dx} - \frac{dU(x^*)}{dx}.$$

Note that

$$g(\tilde{x}) \begin{cases} > 0 & ,\text{if } \tilde{x} < 0 \\ < 0 & ,\text{if } \tilde{x} > 0 \\ = 0 & ,\text{if } \tilde{x} = 0 \end{cases}, \quad (9)$$

due to the fact that $U(x)$ is strictly concave in x, and hence, $(dU(x)/dx)$ is strictly decreasing.

The system (8) can be viewed as a generalized pendulum equation with $g(\tilde{x})$ as the friction term [18]. Let us define a set $\tilde{\Omega}$ as

$$\begin{aligned} \tilde{\Omega} = \{(\tilde{x},\tilde{d}) \in \mathbb{R}^2 : -x^* \le \tilde{x} \le x_{max} - x^* \\ \text{and } -d^* \le \tilde{d} \le d_{max} - d^*\}, \end{aligned}$$

where d_{max} and x_{max} are finite upper-bounds on d and x respectively.

Next define an energy-like Lyapunov function on the set $\tilde{\Omega}$

$$V(\tilde{x},\tilde{d}) = \frac{1}{\alpha}(\tilde{x})^2 + C(\tilde{d})^2\ . \quad (10)$$

Notice that $V(\tilde{x},\tilde{d})$ is positive definite on $\tilde{\Omega}$. The derivative of V along the system trajectories is given by

$$\dot{V}(\tilde{x},\tilde{d}) = \frac{2}{\alpha} g(\tilde{x})\,\tilde{x} \le 0\ ,$$

where the inequality follows from (9). Thus, $\dot{V}(\tilde{x}, \tilde{d})$ is negative semi-definite. Let $S := \{(\tilde{x}, \tilde{d}) \in \tilde{\Omega} : \dot{V}(\tilde{x}, \tilde{d}) = 0\}$. It follows from (9) that $S = \{(\tilde{x}, \tilde{d}) \in \tilde{\Omega} : \tilde{x} = 0\}$. Hence, for any trajectory of the system that belongs to S, we have $\tilde{x} \equiv 0$. It follows then directly from (8) that

$$\tilde{x} \equiv 0 \Rightarrow \dot{\tilde{x}} = 0 \Rightarrow g(\tilde{x}) = 0 \Rightarrow \tilde{d} = 0 \; .$$

Therefore, the only solution that can stay identically in S is the origin, which corresponds to the unique equilibrium of the original system (7). We next consider the effect of boundaries as described by (4) and (6) with $\tilde{d} = d_l$ and $\tilde{x} = x_i$. First, we analyze the case of the unique equilibrium being an inner point. Assume that the trajectory of the system hits the boundary $\tilde{d} = \tilde{d}_{max} := d_{max} - d^* > 0$. In order for the trajectory to stay on this boundary, we need $\dot{\tilde{d}} = \tilde{x}/C \geq 0$. However, we have $\dot{\tilde{x}} < 0$ from (8) as due to (9) $g(\tilde{x}) > 0$ when $\tilde{x} < 0$. Then $\tilde{x}$, and hence $\dot{\tilde{d}}$, necessarily become negative after some time. Thus, the trajectory has to leave this boundary. Furthermore, we have $\dot{V} \leq 0$ on the trajectory of the system. As a result, once the trajectory leaves a boundary it can never hit it again.

We proceed with other three boundaries in a similar fashion. Assume that the trajectory of the system hits the boundary $\tilde{d} = \tilde{d}_{min} := -d^* < 0$. Since from (8) and (9) $\dot{\tilde{x}} > 0$, $\tilde{x}$ and $\dot{\tilde{d}}$ necessarily become positive after some time. Hence, the trajectory has to leave the boundary. On the other hand, when $\tilde{x} = x_{max} - x^*$, we have $g(\tilde{x}) < 0$ and $\dot{\tilde{d}} > 0$. Thus, we obtain $\tilde{d} > 0$ after some delay and $\dot{\tilde{x}} < 0$ from (8), forcing the trajectory out of the boundary. Finally, in the case of $\tilde{x} = -x^* < 0$ we have $g(\tilde{x}) > 0$ and $\dot{\tilde{d}} < 0$. Thus, after some time $\tilde{d} < 0$, and hence $\dot{\tilde{x}} > 0$ from (8). Again, the trajectory leaves the boundary and never returns back due to the non-increasing Lyapunov function V.

In the case of a boundary solution, once the trajectory reaches the equilibrium point it stays on the boundary. For example, assume that $x^* = x_{max} < C$. Then, from (6) we have $\dot{x} = \dot{\tilde{x}} > 0$. Furthermore, $\dot{\tilde{d}} = 0$ from (8). Thus, the trajectory stays on the boundary and on the equilibrium point. In conclusion, the system (7) with boundaries given in (4) and (6) is asymptotically stable on the set $\Omega := \{(x, d) \in \mathbb{R}^2 : 0 \leq x \leq x_{max} \text{ and } 0 \leq d \leq d_{max}\}$ by LaSalle's invariance theorem [18].

B. Stability for a Single Link with Multiple Users

The analysis for a single link with multiple users is a fairly straightforward generalization of the single-link single-user case discussed above. The system has again a unique equilibrium point $(\mathbf{x}^*, d^*)$ [2], at which $(1/\alpha_i)\, dU_i(x_i^*)/dx_i$ is independent of i, $\bar{x}^* = C$ and $d^* = (1/\alpha_i)\, dU_i(x_i^*)/dx_i$. Defining the system around this equilibrium point similar to (8) we obtain

$$\begin{aligned} \dot{\tilde{x}}_i(t) &= g_i(\tilde{x}_i) - \alpha_i \tilde{d}(t) \; , \; i = 1, \ldots, M \\ \dot{\tilde{d}}(t) &= \frac{1}{C} \sum_{i=1}^{M} \tilde{x}_i \end{aligned} \quad , \qquad (11)$$

where $\alpha := [\alpha_1, \ldots, \alpha_M]$ is the user pricing vector, $U_1(x_1), \ldots, U_M(x_M)$ are strictly concave user utility functions, and the functions $g_i(x_i)$ are defined similarily as in the case of (9).

Let us define the generalized set $\tilde{\Omega}$ as

$$\tilde{\Omega} = \{(\tilde{\mathbf{x}}, \tilde{d}) \in \mathbb{R}^{M+1} : -x_i^* \leq \tilde{x}_i \leq x_{i,max} - x_i^* \; , \forall i \\ \text{and } -d^* \leq \tilde{d} \leq d_{max} - d^*\},$$

where d_{max} and $x_{i,max}$ are upper-bounds on d and x_i respectively.

We next define a Lyapunov function on the set $\tilde{\Omega}$, similar to the one of (10):

$$V(\tilde{\mathbf{x}}, \tilde{d}) = \sum_{i=1}^{M} \frac{1}{\alpha_i} (\tilde{x}_i)^2 + C(\tilde{d})^2 \; . \qquad (12)$$

The rest of the analysis is similar to the one in the case of a single link with a single user, and therefore it will not be repeated. In particular, $\dot{V} = \sum_{i=1}^{M} \frac{2}{\alpha_i} g_i(\tilde{x}_i)\tilde{x}_i \leq 0$, and is equal to zero only if $\tilde{x}_i = 0 \, \forall i \Rightarrow \tilde{d} = 0$. Again, the system is asymptotically stable.

C. Stability for a General Network Topology with Multiple Users

We finally establish the stability of the system under a general network topology with multiple links, and with a general routing matrix $\mathbf{A}$ as described in (1). The generalized system is described by (again without the boundary effects)

$$\begin{aligned} \dot{x}_i(t) &= \frac{dU_i(x_i)}{dx_i} - \alpha_i D_i(\mathbf{x}, t) \; , \; i = 1, \ldots, M \\ \dot{d}_l(t) &= \frac{\bar{x}_l}{C_l} - 1 \; , \; l = 1, \ldots, L \end{aligned} \quad , \qquad (13)$$

where $D_i(\mathbf{x}, t) = \sum_{l \in R_i} d_l(\mathbf{x}, t)$, $\bar{x}_l := \sum_{i: l \in R_i} x_i$, and C_l is the capacity of the l^{th} link. For this general case, equilibrium point or points of the system cannot be described explicitly. Therefore, we first investigate the

[2] The proof of uniqueness for a more general case which also captures this special case will be provided in Proposition III.1

uniqueness of the equilibrium. Toward this end, we assume that $\mathbf{A}$ is a full row rank matrix with $M \geq L$ which is in fact no loss of generality as non-bottleneck links on the network have no effect on the equilibrium point, and can be safely left out.

Proposition III.1. *When $\mathbf{A}$ is full row rank, the system (13) has a unique equilibrium point.*

Proof. By setting $\dot{x}_i(t)$ and $\dot{d}_l(t)$ equal to zero for all l and i one obtains

$$\mathbf{A}\,\mathbf{x} = \mathbf{C} \tag{14}$$

$$\mathbf{f}(\alpha, \mathbf{x}) = \mathbf{A}^T \mathbf{d} \;\; , \tag{15}$$

where $\mathbf{d} = [d_1, \ldots, d_L]$ is the delay vector at the links, and the nonlinear vector function $\mathbf{f}$ is defined as

$$\mathbf{f}(\alpha, \mathbf{x}) := \left[\frac{1}{\alpha_i} \frac{dU_i}{dx_i}, \ldots, \frac{1}{\alpha_M} \frac{dU_M}{dx_M} \right].$$

Suppose that there are two different equilibrium points $(\mathbf{x}_1^*, \mathbf{d}_1^*)$ and $(\mathbf{x}_2^*, \mathbf{d}_2^*)$. Then, from (14) it follows that

$$\mathbf{A}\,(\mathbf{x}_1^* - \mathbf{x}_2^*) = 0 \;\Leftrightarrow\; (\mathbf{x}_1^* - \mathbf{x}_2^*)^T \mathbf{A}^T = 0$$

Similarly, from (15) we have

$$\mathbf{f}(\alpha, \mathbf{x}_1^*) - \mathbf{f}(\alpha, \mathbf{x}_2^*) = \mathbf{A}^T (\mathbf{d}_1^* - \mathbf{d}_2^*) \; .$$

Multiplying this with $(\mathbf{x}_1^* - \mathbf{x}_2^*)^T$ from left we obtain

$$(\mathbf{x}_1^* - \mathbf{x}_2^*)^T \left[\mathbf{f}(\alpha, \mathbf{x}_1^*) - \mathbf{f}(\alpha, \mathbf{x}_2^*)\right] = 0$$

We rewrite this as

$$\sum_{i=1}^{M} (\mathbf{x}_{1i}^* - \mathbf{x}_{2i}^*)^T \frac{1}{\alpha_i} \left[\frac{dU_i(x_{1i}^*)}{dx_i} - \frac{dU_i(x_{2i}^*)}{dx_i} \right] = 0.$$

Since U_i's are strictly concave, each term in the summation is negative, with equality holding only if $x_{1i}^* = x_{2i}^*$. Hence, we conclude that x^* has to be unique, that is

$$\mathbf{x}^* = \mathbf{x}_1^* = \mathbf{x}_2^* \;\; .$$

From this, and (13), it immediately follows that $D_i, i = 1, \ldots, M$, are unique. This does not however immediately imply that $d_l, l = 1, \ldots, L$, are also unique. To establish this, we first multiply (15) from left by $\mathbf{A}$,

$$\mathbf{A}\,\mathbf{f}(\alpha, \mathbf{x}^*) = \mathbf{A}\mathbf{A}^T \mathbf{d}$$

Since $\mathbf{A}$ is of full row rank, the square matrix $\mathbf{A}\mathbf{A}^T$ is full rank, and hence invertible. Thus, we obtain a unique $\mathbf{d}^*$ for a given equilibrium flow vector $\mathbf{x}^*$:

$$\mathbf{d}^* = (\mathbf{A}\mathbf{A}^T)^{-1} \mathbf{A}\mathbf{f}(\alpha, \mathbf{x}^*)$$

As a result, $(\mathbf{x}^*, \mathbf{d}^*)$ constitutes a unique equilibrium point for the system (13). □

We note that the unique equilibrium point of the system is only an approximation to the Nash equilibrium since the effect of the i^{th} user on the delay, $D_i(\mathbf{x}, t)$, s/he experiences is ignored. This approximation becomes more accurate as the number of users in the network increases.

Defining the delays at links, d_l, and user flow rates, x_i, around the equilibrium as $\tilde{d}_l := d_l - d_l^*$ and $\tilde{x}_i := x_i - x_i^*$, respectively, for all l and i, we obtain the following system around the equilibrium:

$$\begin{aligned} \dot{\tilde{x}}_i(t) &= g_i(\tilde{x}_i) - \alpha_i \tilde{D}_i(t) \;,\; i = 1, \ldots, M \\ \dot{\tilde{d}}_l(t) &= \frac{1}{C_l} \sum_{i: l \in R_i} \tilde{x}_i \;,\; l = 1, \ldots, L \end{aligned} \quad , \tag{16}$$

where $\tilde{D}_i = \sum_{l \in R_i} \tilde{d}_l$, and $g_i(.)$ is defined as in (9). For the time being, we ignore the effect of boundaries on the system.

Let us define a set $\tilde{\Omega}$ (as before) as

$$\begin{aligned} \tilde{\Omega} = \{(\tilde{\mathbf{x}}, \tilde{\mathbf{d}}) \in \mathbb{R}^{M+L} : &-x_i^* \leq \tilde{x}_i \leq x_{i,max} - x_i^* \\ &\text{and } -d_l^* \leq \tilde{d}_l \leq d_{l,max} - d_l^* \;,\; \forall i \;, l\}, \end{aligned}$$

where $d_{l,max}$ and $x_{i,max}$ are upper bounds on d_l and x_i, respectively.

We next define a Lyapunov function on the set $\tilde{\Omega}$ as a generalized version of the one of (12):

$$V(\tilde{\mathbf{x}}, \tilde{\mathbf{d}}) = \sum_{i=1}^{M} \frac{1}{\alpha_i} (\tilde{x}_i)^2 + \sum_{l=1}^{L} C_l (\tilde{d}_l)^2 \tag{17}$$

The function $V(\tilde{x}, \tilde{d})$ is positive definite on $\tilde{\Omega}$, and its derivative along the system trajectories is given by

$$\dot{V}(\tilde{\mathbf{x}}, \tilde{\mathbf{d}}) = \sum_{i=1}^{M} \frac{2}{\alpha_i} g_i(\tilde{x}_i)\, \tilde{x}_i \leq 0 \;\; ,$$

where the inequality follows because $g_i(\tilde{x}_i)\,\tilde{x}_i \leq 0\, \forall i$. Thus, $\dot{V}(\tilde{\mathbf{x}}, \tilde{\mathbf{d}})$ is negative semidefinite. Let $S := \{(\tilde{\mathbf{x}}, \tilde{\mathbf{d}}) \in \tilde{\Omega} : \dot{V}(\tilde{\mathbf{x}}, \tilde{\mathbf{d}}) = 0\}$. It follows as before that $S = \{(\tilde{\mathbf{x}}, \tilde{\mathbf{d}}) \in \tilde{\Omega} : \tilde{\mathbf{x}} = 0\}$. Hence, for any trajectory of the system that belongs identically to the set S, we have $\tilde{\mathbf{x}} = 0$. It follows directly from (16) that

$$\begin{aligned} \tilde{\mathbf{x}} = 0 \Rightarrow \dot{\tilde{\mathbf{x}}} = 0 \Rightarrow g_i(\tilde{x}) = 0\; \forall i \\ \Rightarrow \tilde{D}_i = 0\; \forall i \Rightarrow \tilde{d}_l = 0\; \forall l, \end{aligned}$$

where the last line is due to the fact that $\tilde{D} = \mathbf{A}^T\, \tilde{\mathbf{d}}^*$ and the matrix $\mathbf{A}$ is of full row rank. Therefore, the only solution that can stay identically in S is the origin, which corresponds to the unique equilibrium of the original system.

We now investigate the effect of the boundaries given in $\tilde{\Omega}$ and described by (4) and (6). First, we analyze the case when the unique equilibrium is not on the boundaries of the set $\tilde{\Omega}$. Consider the case where $\tilde{d}_{\hat{l}} = d_{\hat{l},max} - d^*_{\hat{l}}$ for some link $l = \hat{l}$ while all links except $\hat{l}$ are in equilibrium. Then, for any user i whose flow passes through link $\hat{l}$ and $\tilde{x}_i > 0$, we have $g_i(\tilde{x}_i) < 0$, and from (16) $\dot{\tilde{x}}_i < 0$. Therefore, $\sum_{i:\hat{l}\in R_i} \tilde{x}_i$ decreases until it is negative which in turn makes $\dot{\tilde{d}}_{\hat{l}} < 0$. Thus, the trajectory leaves the boundary. Since $\dot{V} \leq 0$, the trajectory cannot hit the same boundary again. The case $\tilde{d}_l = -d^*_l$ can be handled in a similar fashion. We note that, the case $\tilde{d}_l = -d^*_l$, if it occurs in equilibrium corresponds to an empty buffer at the link l, where the link has no effect on the system for the given set of parameters. As a result, that link can be left out.

For the boundary at $\tilde{x}_i = x_{i,max} - x^*_i$, we have $D_i > 0$ given that all other users passing through links on the path of the i^{th} user are in equilibrium. Then, it immediately follows from (16) that $\dot{\tilde{x}}_i < 0$ and the trajectory leaves the boundary for good. Otherwise, we have a boundary solution with a subset of users transmitting with maximum feasible flow rate, $\tilde{x}_{i,max}$, which contradicts with the initial hypothesis on the equilibrium point. A similar argument holds for the case of $\tilde{x}_i = -x^*_i$, i.e., either there is a boundary solution or the trajectory eventually leaves the boundary and does not hit it again due to the Lyapunov analysis.

We next analyze the case of the equilibrium being on the boundary. Similar to the single user case, once the trajectory reaches the equilibrium point it stays on the boundary. Consider the case where $x^*_i = x_{i,max}$ for the i^{th} user, while other users have equilibrium flows that are less than maximum. Then, from (6) we have $\dot{x}_i = \dot{\tilde{x}}_i > 0$. Furthermore, it follows from (16) that $\dot{d}_l = 0\,,\forall l$. Thus, the trajectory stays on the boundary. We note that the other cases can be handled in a similar fashion. These results are summarized in the following theorem, where we again invoke LaSalle's invariance theorem:

Theorem III.2. *Let* $\mathbf{A}$ *be full row rank. The system*

$$\begin{aligned} \dot{x}_i(t) &= \frac{dU_i(x_i)}{dx_i} - \alpha_i D_i(\mathbf{x}, t)\ ,\ i = 1, \ldots, M \\ \dot{d}_l(t) &= \frac{\bar{x}_l}{C_l} - 1\ ,\ l = 1, \ldots, L \end{aligned}\quad ,$$

with the unique equilibrium point $(\mathbf{x}^*, \mathbf{d}^*)$*, and boundary point behavior described by (4) and (6), is asymptotically stable on the set*

$$\Omega := \{(x, \mathbf{d}) \in \mathbb{R}^{M+L} : 0 \leq x_i \leq x_{i,max} \text{ and } 0 \leq d_l \leq d_{l,max},\ \forall i\ , l\}.$$

IV. Stability under Delay

It was shown in Section III that the system described by (4) and (6) is globally asymptotically stable under a general network topology. We now investigate the global stability of the system under arbitrary propagation delays. First, we analyze the simple case of a single link with a single user to gain insight into the problem. Next, we generalize the analysis to a general network with a single bottleneck node and multiple users.

A. Stability for a Single Link with a Single User under Delay

For the case of a single user on a single link, we modify equation (7) describing the system around the equilibrium by introducing a maximum delay r between the user and the link:

$$\begin{aligned} \dot{\tilde{x}}(t) &= g(\tilde{x}) - \alpha \tilde{d}(t - r) \\ \dot{\tilde{d}}(t) &= \frac{1}{C}\tilde{x}(t - r) \end{aligned}\quad . \tag{18}$$

Notice that (18) is a set of delay differential equations. Such systems have been studied extensively in [19], [20]. Here we will particularly make use of the methods presented in Chapter 4.2 of [20]. From (18), we immediately have

$$\dot{\tilde{x}}(t) = g(\tilde{x}(t)) - \alpha\tilde{d}(t + r) + \alpha[\tilde{d}(t + r) - \tilde{d}(t - r)],$$

and

$$\dot{\tilde{x}}(t - r) = g(\tilde{x}(t - r)) - \alpha\tilde{d}(t) + \frac{\alpha}{C}\int_{-2r}^{0} \tilde{x}(t + s)ds\,.$$

On the same set $\tilde{\Omega}$ as in the delay-free case, we define a Lyapunov function

$$\begin{aligned} V(\tilde{x}, \tilde{d}) = \ & \frac{1}{\alpha}(\tilde{x}(t - r))^2 + C(\tilde{d}(t))^2 \\ & + \frac{1}{C}\int_{-2r}^{0}\int_{t+s}^{t} \tilde{x}^2(u - r)du\ ds \end{aligned}\quad , \tag{19}$$

which is positive definite in $\tilde{\Omega}$. Taking the derivative of V along the system trajectories, we obtain

$$\begin{aligned} \dot{V}(\tilde{x}, \tilde{d}) = \ & \frac{2}{\alpha}g(\tilde{x}(t - r))\tilde{x}(t - r) \\ & + \frac{2}{C}\int_{-2r}^{0} \tilde{x}(t - r)\tilde{x}(t + s - r)ds \\ & + \frac{1}{C}\int_{-2r}^{0} [\tilde{x}^2(t - r) - \tilde{x}^2(t + s - r)]ds \end{aligned}$$

Using the simple algebraic inequality

$$2\tilde{x}(t - r)\tilde{x}(t + s - r) \leq \tilde{x}^2(t - r) + \tilde{x}^2(t + s - r),$$

one can bound the derivative $\dot{V}$ above by

$$\dot{V}(\tilde{x},\tilde{d}) \leq \frac{2}{\alpha} g(\tilde{x}(t-r))\tilde{x}(t-r) + \frac{4r}{C}\tilde{x}^2(t-r)$$

Thus, $\dot{V}(\tilde{x},\tilde{d})$ can be made negative semi-definite by imposing a condition on the maximum delay r. In this case, let $S := \{(\tilde{x},\tilde{d}) \in \tilde{\Omega} : \dot{V}(\tilde{x},\tilde{d}) = 0\}$. It follows from (9) that $S = \{(\tilde{x},\tilde{d}) \in \tilde{\Omega} : \tilde{x} = 0\}$. Hence, for any trajectory of the system that belongs to S, we have $\tilde{x} \equiv 0$. It also follows directly from (18) that

$$\tilde{x} \equiv 0 \Rightarrow \dot{\tilde{x}} = 0 \Rightarrow g(\tilde{x}) = 0 \Rightarrow \tilde{d} = 0 \; .$$

Therefore, the only solution that can stay identically in S is the origin, which corresponds to the unique equilibrium of the original system (7).

We thus conclude that the system (18) is asymptotically stable by LaSalle's invariance theroem if the maximum delay r satisfies the condition

$$r < \frac{C}{2\alpha} k \, , \tag{20}$$

where k is defined as

$$k := \inf_{-x^* \leq \tilde{x} \leq x_{max} - x^*} \left| \frac{g(\tilde{x})}{\tilde{x}} \right| .$$

In order to gain further insight into this condition, we compute the parameter k for the specific case when the utility function is taken as the logarithmic one, that is $U(x) = u \log(x+1)$. In this case we obtain

$$g(\tilde{x}) = \frac{u}{x+1} - \frac{u}{x^*+1} = \frac{-u\tilde{x}}{(x+1)(x^*+1)}$$

and hence

$$k = \min_{0 \leq x \leq x_{max}} \frac{u}{(x+1)(x^*+1)} = \frac{u}{(x_{max}+1)(x^*+1)} .$$

The requirement on the delay term r is dependent on the equilibrium x^*, and since $x^* \in [0, x_{max}]$, a safe bound on r is

$$r < \frac{uC}{2\alpha(x_{max}+1)^2} .$$

Of course a better bound can be obtained on r if we know that $x^* \ll x_{max}$, and that the trajectory also remains in a small neighborhood of the equilibrium, x^*. This would very much be dependent on the application at hand.

The analysis of the effect of boundaries on the system is almost identical to the one of the case without delay. Assume that $\tilde{d}(t) = \tilde{d}_{max} \; \forall t \in [-r, 0]$. In order for the trajectory to stay on the boundary at $t > 0$, one needs $\tilde{x}(t) > 0 \; \forall t \in [-r, 0]$. However, we have $\dot{\tilde{x}}(t) < 0$ from (18). Hence, after some time, $\dot{\tilde{d}} < 0$, and the trajectory leaves the boundary. Since $\dot{V} < 0$ the system trajectory may never return to the boundary. The analysis of the remaining boundaries can be handled in a similar way, and will be omitted. This now brings us to the following theorem, where again LaSalle's invariance theorem is invoked:

Theorem IV.1. *The system*

$$\begin{aligned} \dot{x}(t) &= \frac{dU(x(t))}{dx} - \alpha d(t-r) \\ \dot{d}(t) &= \frac{1}{C} x(t-r) - 1 \end{aligned} \; ,$$

with the unique equilibrium point $(\mathbf{x}^*, \mathbf{d}^*)$ *and boundary point behavior described by (4) and (6) is asymptotically stable on the set* Ω *if the maximum delay,* r*, in the system satisfies the condition*

$$r < \frac{kC}{2\alpha} \, ,$$

where $k := \inf_{-x^* \leq \tilde{x} \leq x_{max} - x^*} |g(\tilde{x})/\tilde{x}|$.

B. Stability for a Single (Bottleneck) Link with Multiple Users under Delay

We now generalize the preceding analysis of a single link with a single user to multiple users by introducing user specific maximum delays $r = [r_1, \ldots, r_M]$ between the link and the users. The system has a unique equilibrium point $(\mathbf{x}^*, d^*)$ as characterized in Section III. Modifying the system equations (11) around this equilibrium point by introducing the associated maximum delays we obtain

$$\begin{aligned} \dot{\tilde{x}}_i(t) &= g_i(\tilde{x}_i(t)) - \alpha_i \tilde{d}(t - r_i) \; , \; i = 1, \ldots, M \\ \dot{\tilde{d}}(t) &= \frac{1}{C} \sum_{i=1}^{M} \tilde{x}_i(t - r_i) \end{aligned} \; . \tag{21}$$

Following an approach similar to the one in the single user case one gets for the i^{th} user

$$\begin{aligned} \dot{\tilde{x}}_i(t - r_i) = \; & g_i(\tilde{x}_i(t - r_i)) - \alpha_i \tilde{d}(t) \\ & + \frac{\alpha_i}{C} \int_{-2r_i}^{0} \sum_{j=1}^{M} \tilde{x}_j(t + s - r_j) ds. \end{aligned}$$

We again define a positive definite Lyapunov function on the same corresponding set $\tilde{\Omega}$ as in the delay-free case:

$$\begin{aligned} V(\tilde{\mathbf{x}}, \tilde{d}) = \; & \sum_{i=1}^{M} \frac{1}{\alpha_i} (\tilde{x}_i(t - r_i))^2 + C(\tilde{d}(t))^2 \\ & + \frac{M}{C} \sum_{i=1}^{M} \int_{-2r_i}^{0} \int_{t+s}^{t} \tilde{x}_i^2(u - r_i) du \; ds. \end{aligned} \tag{22}$$

Taking the derivative of V along the system trajectories, we obtain

$$\begin{aligned}\dot{V}(\tilde{\mathbf{x}},\tilde{d}) &= \sum_{i=1}^{M} \frac{2}{\alpha_i} g_i(\tilde{x}_i(t-r_i))\tilde{x}(t-r_i) \\ &+\frac{1}{C}\int_{-2r_i}^{0} \sum_{i=1}^{M}\sum_{j=1}^{M} 2\tilde{x}_i(t-r_i)\tilde{x}_j(t+s-r_j)ds \\ &+\frac{M}{C}\sum_{i=1}^{M}\int_{-2r_i}^{0}[\tilde{x}_i^2(t-r)-\tilde{x}_i^2(t+s-r)]ds\end{aligned}$$

We bound the derivative $\dot{V}$ above by

$$\begin{aligned}\dot{V}(\tilde{\mathbf{x}},\tilde{d}) \leq \quad & \sum_{i=1}^{M} \frac{2}{\alpha_i} g_i(\tilde{x}_i(t-r_i))\tilde{x}_i(t-r_i) \\ &+\frac{4Mr_i}{C}\tilde{x}_i^2(t-r_i)\end{aligned}$$

The derivative of V can be made strictly negative by imposing a condition on the maximum delay in the system, $r_{max} := \max_i r_i$. In this case, let $S := \{(\tilde{\mathbf{x}},\tilde{d}) \in \tilde{\Omega} : \dot{V}(\tilde{\mathbf{x}},\tilde{d}) = 0\}$. It follows as before that $S = \{(\tilde{\mathbf{x}},\tilde{d}) \in \tilde{\Omega} : \tilde{\mathbf{x}} = 0\}$. Hence, for any trajectory of the system that belongs identically to the set S, we have $\tilde{\mathbf{x}} = 0$. It also follows directly from (21) that

$$\tilde{\mathbf{x}} = 0 \Rightarrow \dot{\tilde{\mathbf{x}}} = 0 \Rightarrow g_i(\tilde{x}) = 0 \;\forall i \Rightarrow \tilde{d} = 0,$$

where we have made use of the fact that the matrix $\mathbf{A}$ is of full row rank. Therefore, the only solution that can stay identically in S is the origin, which corresponds to the unique equilibrium of the original system. As a result, the system (21) is asymptotically stable by LaSalle's invariance theorem if the maximum delay in the system, r_{max}, satisfies the condition

$$r_{max} < \frac{k_{min}C}{2M\alpha_{max}}\,, \tag{23}$$

where α_{max} and k_{min} are defined as

$$\begin{aligned}\alpha_{max} &:= \max_i \alpha_i \\ k_{min} &:= \min_i \inf_{-x_i^* \leq \tilde{x}_i \leq x_{i,max}-x_i^*} \left|\frac{g(\tilde{x}_i)}{\tilde{x}_i}\right| \,.\end{aligned} \tag{24}$$

Notice that the bound on the maximum delay required for the stability in the system is affected by, among other things, the maximum pricing parameter and the number of users.

We investigate the effect of boundaries on the system first in the case of the equilibrium being an inner one. Consider the case $\tilde{d}(t) = d_{max} - d^* \;\forall t \in [-r, 0]$. Then, for any user i with $\tilde{x}_i(t) > 0$, we have $g_i(\tilde{x}_i(t)) < 0$, and from (21) $\dot{\tilde{x}}_i(t) < 0$, $t > 0$. Therefore, $\sum_i \tilde{x}_i$ decreases until it is negative which in turn makes $\dot{\tilde{d}} < 0$. Thus, the trajectory leaves the boundary. Since $\dot{V} < 0$, the trajectory cannot hit the same boundary again. A similar analysis also applies to the case $\tilde{d} = -d^*$. For boundaries on $\tilde{x}_i$, assume that all users but the i^{th} one are in equilibrium, and $\tilde{x}_i = x_{i,max} - x_i^* \;\forall t \in [-r, 0]$. Then, we necessarily have $\tilde{d} > 0$ after some time, and hence $\dot{\tilde{x}}_i < 0$. Thus, the trajectory leaves the boundary, and never returns due to the strictly decreasing Lyapunov function V. Similar arguments also hold for the case when all users but the i^{th} one are in equilibrium, and $\tilde{x}_i = -x_i^*$. In the case of boundary solutions, the analysis is identical to earlier ones and therefore will be omitted. The following theorem now extends the results of Theorem IV.1 to the multi-user case.

Theorem IV.2. *The system*

$$\begin{aligned}\dot{x}_i(t) &= \frac{dU_i(x_i(t))}{dx_i} - \alpha_i d(\mathbf{x}, t-r)\;,\; i = 1,\ldots,M \\ \dot{d}(t) &= \frac{1}{C}\sum_{i=1}^{M} x_i(t-r_i) - 1\end{aligned}\,,$$

with the unique equilibrium point $(\mathbf{x}^*, d^*)$, *and boundary point behavior described by (4) and (6), is asymptotically stable on the corresponding set* Ω, *if the maximum delay,* r_{max}, *in the system satisfies the condition*

$$r_{max} < \frac{k_{min}C}{2M\alpha_{max}}\,,$$

where α_{max} *and* k_{min} *are defined in (24).*

V. An Implementation of the Congestion Control Scheme

The continuous time network model, cost function and user responses in Section II are based on fluid approximations. In reality, however, users update their flow rates only at discrete time instances corresponding to multiples of RTT. Hence, for implementation purposes, we discretize the reaction function of the i^{th} user, and normalize it with respect to the RTT of the user. In addition, we need a specific utility function in order to quantify the user response in (6). Logarithmic utility functions are widely used in the literature not only because they have nice properties like strict concavity but also because they adequately capture several important concepts economics, such as the law of diminishing returns. We choose the following utility function for i^{th} user:

$$U_i(x_i) = u_i \log(x_i + 1),$$

where u_i is a user specific utility parameter. The optimal user response is, therefore, a discretized version of (6), and is given by

$$x_i(t+1) = \left[x_i(t) + \kappa_i \left[\frac{u_i}{x_i(t)+1} - \alpha_i \sum_{l \in R_i} d_l(t) \right] \right]^+, \quad (25)$$

where κ_i is a (user specific) step-size constant.

The congestion control scheme characterized by the user response (25) is implemented in a Game (theory) Based Congestion Control (GBCC) protocol using the Network Simulator 2 (*ns-2*) [21]. The simulator *ns-2* is chosen because it provides both a realistic environment for testing the proposed congestion control scheme and a level of abstraction for easy implementation. GBCC is a simple window based protocol for best-effort data traffic. It is devised as an end-to-end sliding window protocol [22], where the sender side adjusts its window size according to the reaction function (25). For simplicity, receiver window size is chosen as one. We also implement a version with a simple slow start mechanism where the window size is increased by one per RTT until a packet loss is observed. We next give an overview on GBCC scheme by summarizing the sender and receiver side functionalities.

A. GBCC Protocol

As one of the goals of GBCC protocol is compatibility with existing protocols, most of the functionality is on the sender side. Specifically, the sender side has the following functions:

- The sender puts sequence number and time stamp into the packet header. It estimates RTT and base RTT, which is calculated as the minimum of the RTTs until that moment, by using the received acknowledgment (ack) packets. The estimation method for RTT is the same as the one in [23].
- If a double ack is received, i.e. the same packet is acknowledged twice by the receiver, then it retransmits the packages beginning from the last acknowledged packet number. We note that, this *go back n* scheme [22] is implemented for its simplicity. In fact, better mechanisms with receiver window size being larger than one exist.
- The sender updates the window size according to (25) using the current value of queueing delay, which is taken as the difference between the current RTT and base RTT. The window size, $W > 0$, is strictly positive.
- If no ack packet is received within, say $2\,RTT$, then sender retransmits previous packets beginning from the last acknowledged one, and reduces the window size.

The receiver side, on the other hand, has the function of acknowledging received packets. If no packet is received for a specific time, say $4\,RTT$, last received packet is acknowledged again.

VI. Simulations

We simulate the proposed congestion control scheme, GBCC, on *ns-2*. The underlying protocol used for routing is the standard IP. Links and queues are chosen to be duplex and drop-tail, respectively. For simplicity, we fix the packet size to $1,000\ bytes$. First, we simulate GBCC without a slow start mechanism in the simple single-user single-link case. The parameters in (25) are chosen as $\alpha = 30$ and $u = 10,000$. The buffer size is $50KB$ and RTT varies from $10ms$ to $50ms$, and to $200ms$. We observe in Figure 1 that as RTT gets too large, the system becomes unstable in accordance with the analysis in Section IV. Notice that it takes up to 7 seconds for the flow to reach its capacity in this simulation. Therefore, we use the slow start version of GBCC for the rest of the study.

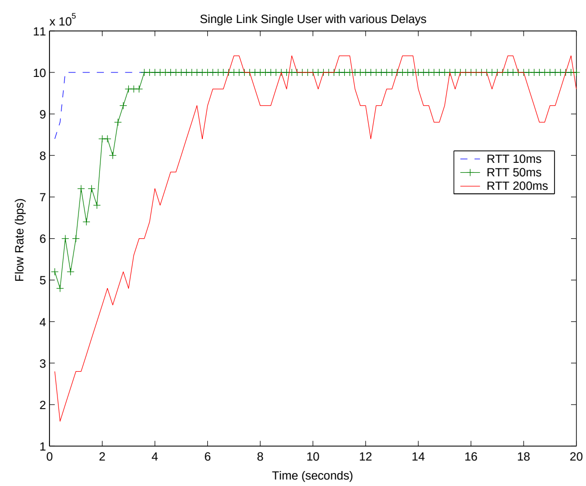

Fig. 1. A single user on a single link with $RTT = 10, 50,$ and $200ms$. This version of GBCC has no slow start mechanism.

We next explore the interaction between the GBCC and TCP on a single bottleneck link with $10ms$ delay. GBCC is TCP-friendly as it can be observed from Figure 2. The fluctuation in the first two seconds is due to the slow start mechanism which requires a packet loss for termination. In the final simulation on a single bottleneck link, there are 20 identical users with parameters $\alpha = 50$, $u = 400,000$, and delays are randomly chosen between $2ms$ and $30ms$ according to a uniform distribution. We observe flows of 3 specific users with respective delays of $2ms$, $15ms$, and $50ms$ in Figure 3. The system again converges to the

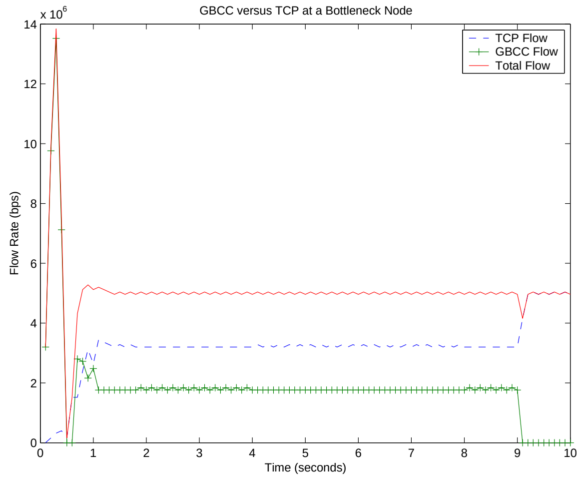

Fig. 2. GBCC flow versus TCP flow on a bottleneck link with $10ms$ delay.

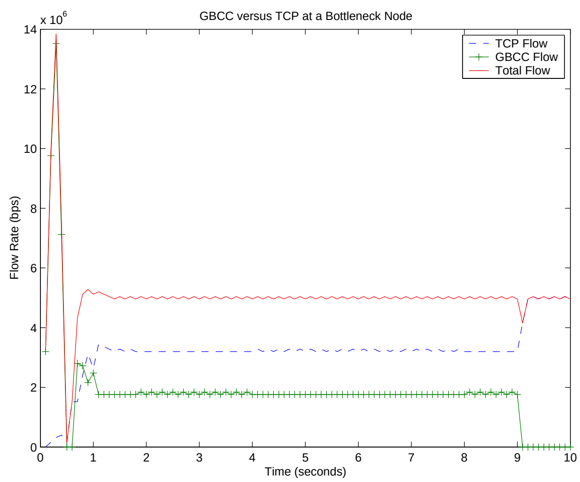

Fig. 3. Three out of 20 flows with various delays ($2ms$, $15ms$, and $50ms$) sharing a $5Mbps$ bottleneck link.

equilibrium, however similar to TCP, GBCC favors flows with smaller RTT as it is a window-based scheme.

The following simulation is done with three users on a simple three node network topology with two $5Mbps$ links of $20ms$ delay as shown in Figure 4. While flows of user 1 and 2 pass through links 1 and 2 respectively, the flow of user 3 passes through both links. Cost parameters are chosen as $\alpha = 30$ and $u = 400,000$. User 3 is 'charged' more than others through summation of queueing delays as s/he uses resources on both links. Thus, having the same utility parameter as others, s/he obtains a smaller fraction of the bandwidth. Figure 5 depicts the flow rates of user 2 and 3 as observed in node 2.

Finally, we simulate 10 users with various routes and experiencing various delays on a seven node arbitrary topology network (Figure 6) with all links except the one between nodes 5 and 6 having capacity of $5Mbps$ each. The link between nodes 5 and 6, on the other hand, has

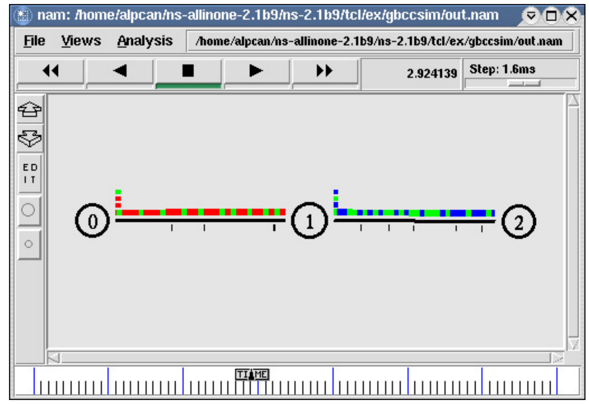

Fig. 4. A *Nam* screenshot of the simple network. Links are symmetric, and have a capacity of $5Mbps$ with $20ms$ delay.

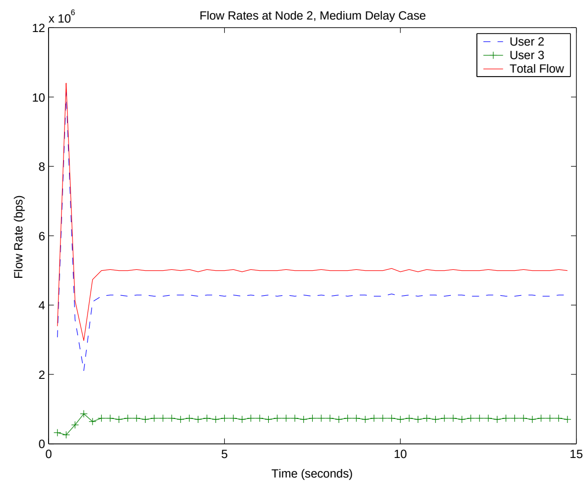

Fig. 5. Flows of users 2, 3 and total flow at node 2 are observed for 15 seconds.

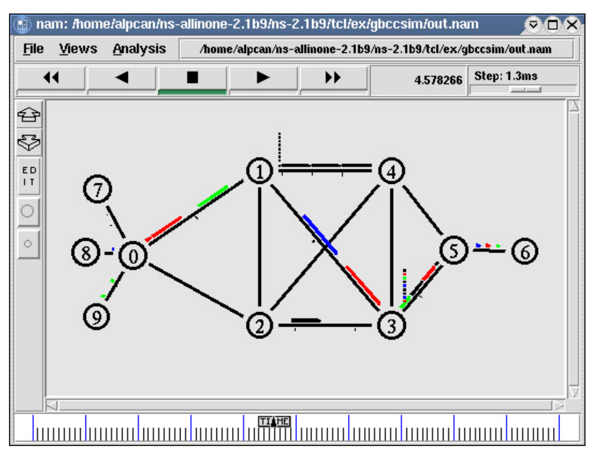

Fig. 6. A *Nam* screenshot of the general (arbitrary) topology network.

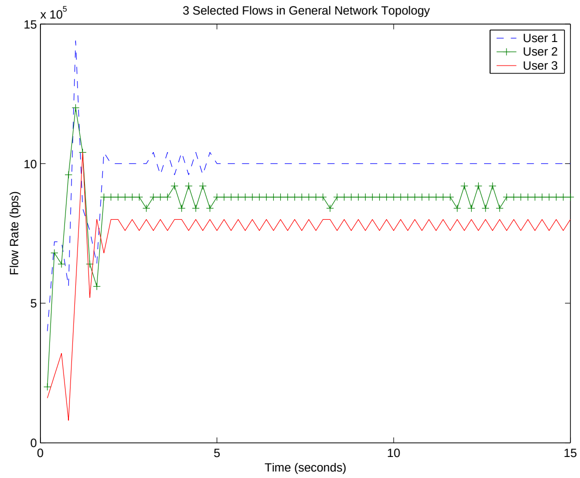

Fig. 7. Three flows from nodes 7, 8, and 9 to node 6 are shown where these users are symmetric and have the following cost parameters: $\alpha = 30$ and $u = 200,000$.

a capacity of $10Mbps$. The links have equal propagation delays of $5ms$ each, except the links to nodes 7, 8, and 9, which have delays of $5ms$, $10ms$, and $25ms$, respectively. The users at nodes 7, 8, and 9 all have connections to node 6 and each experiences a different propagation delay. Figure 7 shows only the flows of these three users as measured at node 6. We note that although the number of links in this simulation is equal to the number of users, the number of bottleneck links that affect the equilibrium flows is actually smaller. Hence, the routing matrix $\mathbf{A}$ is of full row rank.

VII. Conclusion

In this paper, we have developed and analyzed a congestion control game with a linear pricing scheme based on variations in the queueing delay experienced by the users. User demand for bandwidth is captured by a broad class of utility functions that are strictly increasing and strictly concave. The objective function for each user in this noncooperative game is defined as the difference between the pricing and utility functions. Using a network model based on fluid approximations and through a realistic modelling of queues in the network, we have established the existence of a unique equilibrium, and the global stability of the equilibrium point for a general network topology. We have also provided sufficient conditions for system stability on a bottleneck link shared by multiple users under non-negligible propagation delays.

We have implemented and simulated a simple, window-based, end-to-end congestion control scheme in *ns-2* network simulator based on the theoretical foundations of the congestion control game. We have investigated several properties of the scheme developed through simulations on a single bottleneck link and on various general network topologies with non-negligible propagation delays. These simulations reveal that the implemented scheme not only confirms the theoretical results but is also TCP-friendly.

There still remain a few open issues and many directions for future research. For example, there is still ample room for improvement in the implementation of the congestion control scheme, such as increasing the receiver window size and fine tuning the slow start mechanism. Another topic for further study would be to devise a methodology for choosing the pricing parameter α. Yet another direction for future research would be the derivation of improved (less restrictive) sufficient conditions on the maximum delay allowable in a general network, to ensure stability of the overall system.

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